CONTROLLED BOUNDARY AND h-COBORDISM THEOREMS

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ABSTRACT. In this paper two theorems are established which are consequences of some earlier approximation results of the author. The first theorem is a controlled boundary theorem for finite-dimensional manifolds. By this we mean an ordinary boundary theorem plus small ε-control in a given parameter space. The second theorem is a controlled h-cobordism theorem for finite-dimensional manifolds with small ε-control in a given parameter space. These results generalize the End Theorem and the Thin h-Corbordism of Quinn.

1. Introduction. The purpose of this paper is to establish two theorems which are consequences of the approximation results of [2]. The first theorem is a controlled boundary theorem for finite-dimensional manifolds. By this we mean an ordinary boundary theorem, as treated in Siebenmann [8], plus small ε-control in a given parameter space. The second theorem is a controlled h-cobordism theorem for finite-dimensional manifolds with similar small ε-control in a given parameter space. These results generalize the End Theorem and the Thin h-Cobordism Theorem of Quinn [7] in the sense that they do not require the parameter space to be locally simply connected and they do not require that the map to the parameter space be locally constant on π_1. This is accomplished by replacing the stable algebra (i.e. geometric module) approach of Quinn [7, §8] by the more topological approximation results of [2].

Thoughout this paper B will always be a finite-dimensional compact metric space. If M^m is a noncompact (TOP or PL) manifold and $p: M \to B$ is a map, then we say that M admits a *controlled boundary* if p continuously extends to $\tilde{p}: \tilde{M} \to B$, where $\tilde{M}$ is a compact manifold and $M = \tilde{M} - \partial\tilde{M}$. Our Controlled Boundary Theorem below imposes three conditions on M in order for it to admit such a controlled boundary. The first two are necessary, while the third (a local π_1 requirement) is not. We first describe the necessary conditions.

We say that $p: M \to B$ is *tame at* ∞ provided that given any $\varepsilon > 0$ and compactum $C \subset M$ there exists a larger compactum $D \supset C$ and a homotopy $h_t: M - C \to M - C$ such that $h_0 = \mathrm{id}$, $h_1(M - C) \subset D - C$, and $ph_t: M - C \to B$ is an ε-homotopy. This latter requirement means that the diameter of the track of each point under this homotopy is less than ε. It is obvious that if M admits a controlled boundary, then p must be tame at ∞. The second necessary condition is motivated by the condition "π_1 stable at ∞" that was used in Siebenmann [8] to describe the

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uncontrolled problem. Loosely phrased it means that fundamental groups of complements of large compacta in M can be moved toward ∞ with small ε-control in B. Specifically we define $p : M \to B$ to be 1-*movable at* ∞ provided that given any $\varepsilon > 0$ and compactum $C \subset M$ there exists a larger compactum $D \supset C$ so that the following property is satisfied: for every compactum $E \supset D$ there exists a larger compactum $F \supset E$ such that if (K, L) is any 2-dimensional compact polyhedral pair and $f : (K, L) \to (M - D, M - F)$ is a map, then there exists a map $g : (K, L) \to (M - E, M - F)$ such that:

   (1) $g = f$ on L,

   (2) pg is ε-close to pf,

   (3) if K^1 is the 1-skeleton of K, then there is a homotopy $h_t : f|K^1 \cup L \simeq g|K^1 \cup L$ rel L in $X - C$ for which ph_t is an ε-homotopy.

Observe that if we ignore the control in B and assume that everything is connected, then the above properties imply that

$$\text{image}\big(\pi_1(M - E) \to \pi_1(M - C)\big) = \text{image}\big(\pi_1(M - D) \to \pi_1(M - C)\big)$$

and

$$\text{kernel}\big(\pi_1(M - F) \to \pi_1(M - D)\big) = \text{kernel}\big(\pi_1(M - F) \to \pi_1(M - E)\big).$$

Again it is obvious that if M admits a controlled boundary, then p is 1-movable at ∞.

In the sequel Wh will denote the Whitehead group functor, which is a homotopy functor from the category of spaces and homotopy classes of maps to the category of abelian groups and homomorphisms [5, §10]. By direct summing over components this makes good sense even if the spaces are not path connected. We say that $p : M \to B$ is *nice at* ∞ provided that for every $b \in B$, neighborhood $U \subset B$ of b, and compactum $C \subset M$, there exists a smaller neighborhood $V \subset U$ of b and a larger compactum $D \supset C$ for which the inclusion

$$\big[p^{-1}(V) \cap (M - D) \big] \times T^k \hookrightarrow \big[p^{-1}(U) \cap (M - C) \big] \times T^k$$

induces the 0-homomorphism on Whitehead groups, for all k-tori T^k. (We also allow $p^{-1}(V)$ and $p^{-1}(U)$ to be empty.) A simple example of this is provided by a map $p : M \to B$ which is UV^1 *at* ∞, i.e., for any $b \in B$, neighborhood $U \subset B$ of b, and compactum $C \subset M$, there exists a neighborhood $V \subset U$ of b and a compactum $D \supset C$ for which any loop in $p^{-1}(V) \cap (M - D)$ is null-homotopic in $p^{-1}(U) \cap (M - C)$. Such a map is easily seen to be nice at ∞ by using the dependence of the Whitehead group functor on π_1, the homotopy functorality of Wh, and the difficult calculation $\text{Wh}(T^k) = 0$ [1]. Of course, niceness at ∞ is not a necessary condition for M to admit a controlled boundary. It is just a condition imposed on M which is used to conclude that certain obstructions to putting a controlled boundary on M vanish.

CONTROLLED BOUNDARY THEOREM. *Let M^m be a manifold, $m \geqslant 6$ and $\partial M = \varnothing$, and let $p : M \to B$ be a map which is tame at ∞, 1-movable at ∞, and nice at ∞. Then M admits a controlled boundary.*

REMARKS. There are also the expected relative versions of this result which are similar to those stated in the Relative End Theorem of [7]. For example if $\partial M \neq \varnothing$ and ∂M already has a controlled boundary, then M can be given a controlled boundary which agrees with the given one on ∂M. Also if $C \subset B$ is compact and the portion of M over some neighborhood of C already has a controlled boundary, then M can be given a controlled boundary which agrees with the given one over C.

In order to state our Controlled h-Cobordism Theorem we will need more definitions. If W^{m+1} is a compact (TOP or PL) manifold and $p : W \to B$ is a map, then a cobordism $(W, \partial_0 W)$ is said to be an (ε, h)-*cobordism* provided that there are deformation retractions of W to $\partial_0 W$ and $\overline{\partial W - \partial_0 W}$ so that the images of these deformations in B are ε-homotopies. Similarly we say that $(W, \partial_0 W)$ is an ε-*product* if there exists a homeomorphism $h : [0, 1] \times \partial_0 W \to W$ for which $h(0, x) = x$ and ph is ε-close to the composition

$$p(\text{proj}) : [0, 1] \times \partial_0 W \overset{\text{proj}}{\to} \partial_0 W \overset{p}{\to} B.$$

For a given $\varepsilon > 0$ our Controlled h-Cobordism Theorem below imposes conditions which guarantee that there is a small $\delta > 0$ for which a (δ, h)-cobordism becomes an ε-product. Here is the basic definition. For any $b \in B$ let $S_\varepsilon(b)$ be the open ε-ball around b. Then we say that $p : W \to B$ is (ε, δ)-*nice* if given any $b \in B$ the inclusion-induced homomorphism,

$$\text{Wh}\left(p^{-1}(S_\delta(b)) \times T^k \right) \to \text{Wh}\left(p^{-1}(S_\varepsilon(b)) \times T^k \right),$$

is the 0-homomorphism, for all k. A simple example of this is provided by $p = \text{proj} : B \times F \to B$, where B and F are compact ANR's for which $\pi_1(C)$ is free abelian for all path components C of F. From the π_1-condition we conclude that $\text{Wh}(F \times T^k) = 0$, and from the local contractibility of B (and the homotopy functorality of Wh) we see that for every $\varepsilon > 0$ there exists a $\delta > 0$ for which p is (ε, δ)-nice.

CONTROLLED h-COBORDISM THEOREM. *Suppose we are given a compactum B, an integer $m \geq 5$, and $\varepsilon > 0$. There exists a decreasing set $\{\delta_i\}_{i=1}^k$ of positive numbers so that if W^{m+1} is a manifold, $p : W \to B$ is a map which is (δ_i, δ_{i+1})-nice for all i, and $(W, \partial_0 W)$ is a (δ_k, h)-cobordism, then $(W, \partial_0 W)$ is an ε-product.*

The scope of this result is greatly enhanced by the following addendum which clarifies the manner in which k and the δ_i are chosen.

ADDENDUM. 1. *The integer k depends only on* dim B.

2. *The set $\{\delta_i\}$ clearly depends on B, m, and ε, and it can be rechosen as follows: For a fixed i_0 and any $x \in (0, \delta_{i_0})$ we can find another set $\{\delta_i'\}_{i=1}^k$, fulfilling the above requirements, which is of the form $\delta_i' = \delta_i$, for $i < i_0$, and $\delta_{i_0}' = x$.*

REMARK. As in the case of the Controlled Boundary Theorem there is also the expected relative version of this result when $(W, \partial_0 W)$ has a given controlled product structure over a portion of B.

Concerning the proofs of the Controlled Boundary Theorem and the Controlled h-Cobordism Theorem we will only give the details for the PL case. As is usually

done, the TOP case is dealt with by merely reminding the reader that, in the dimension ranges we are concerned with, the appropriate handlebody techniques exist to translate our PL arguments into the TOP category [6]. So in the sequel we will be working only in the PL category.

Finally we make a few remarks concerning the organization of the material in this paper. In §3 we establish a controlled Hurewicz theorem which is needed throughout the paper. This is more general than a similar result of [7], for it is designed to specifically apply to our situation in which there is no local control on π_1. §4 contains some notation and observations concerning controlled matrices. It is merely a formulation of the algebraic language needed in §5, which deals with a controlled version of Wall's finiteness obstruction theory. This is complicated by the fact that multiple basepoints must be dealt with simultaneously. This is all put together in §6 for a proof of the Controlled Boundary Theorem. In §7 we prove the Controlled h-Cobordism Theorem by using the method of proof of the Controlled Boundary Theorem to split up a given controlled h-cobordism into a number of small h-cobordisms. In order to pull this off we have to invoke a relative version of the Approximation Theorem of [2].

2. Definitions and notation. The purpose of this section is to introduce some language which will be used throughout this paper. First a word about the use of the term *clean*. If X is a polyhedron, then a closed subset A of X is said to be *clean* in X provided that A is a subpolyhedron and the topological boundary of A, Bd(A), is a subpolyhedron which is PL bicollared in X. This means that Bd(A) has a PL collaring in both A and the closure of $X - A$, $\overline{X - A}$. If M^m is a manifold, possibly with boundary, then a closed subset N^n of M is *clean* provided that N is a PL submanifold and Bd(N) is a PL submanifold which is PL bicollared. This necessarily means that $N \cap \partial M$ is also clean in ∂M.

Next we give some definitions which generalize to the controlled setting some standard notions from the homotopy category. In what follows $p: Y \to B$ will be a controlling map from a space Y to our parameter space B. For convenience we say that a homotopy into Y is a $p^{-1}(\varepsilon)$-*homotopy* provided that the composition of the homotopy with p is an ε-homotopy into B. Our first definition is a controlled version of the notion of a homotopy equivalence. We say that a map $f: X \to Y$ is an ε-*equivalence* provided that there is a map $g: Y \to X$ for which fg is $p^{-1}(\varepsilon)$-homotopic to id$_Y$ and gf is $(pf)^{-1}(\varepsilon)$-homotopic to id$_X$. Now generalizing the notion of an n-equivalence we say that $f: X \to Y$ is an (ε, n)-*equivalence* if for every $k \leqslant n + 1$ and commutative diagram

$$
\begin{array}{ccc}
\partial I^k & \xrightarrow{\phi_0} & X \\
\uparrow & & \downarrow f \\
I^k & \xrightarrow{\phi} & Y
\end{array}
$$

there exists a map $\tilde{\phi}: I^k \to X$ such that $\tilde{\phi}|\partial I^k = \phi_0$, $d(pf\tilde{\phi}, p\phi) < \varepsilon$, and for each $k \leqslant n$ there exists a $p^{-1}(\varepsilon)$-homotopy $f\tilde{\phi} \simeq \phi$ rel ∂I^k. Finally generalizing the notion of an n-connected map we say that $f: X \to Y$ is (ε, n)-*connected* if for every $k \leqslant n$

and commutative square as above, there exists a map $\tilde{\phi} : I^k \to X$ such that $\tilde{\phi} \,|\, \partial I^k = \phi_0$ and there exists a $p^{-1}(\varepsilon)$-homotopy $f\tilde{\phi} \simeq \phi$ rel ∂I^k.

We will also have to have a notion of spaces which are homotopically dominated by compact polyhedra with ε-control in B. If $p : Y \to B$ is a controlling map as above, then we define Y to be ε-*finitely dominated* provided that there exist a compact polyhedron K and maps $f : K \to Y$, $g : Y \to K$ such that fg is $p^{-1}(\varepsilon)$-homotopic to id_Y. The map f is called an ε-*domination* and the map g is called a (right) ε-*inverse* of f.

3. A controlled Hurewicz Theorem. Throughout this section (X, Y) will be a pair of locally compact separable metric ANR's and $p : X \to B$ will be a controlling map to our parameter space. We also assume that Y is compact and X is ε-finitely dominated, for all $\varepsilon > 0$.[2] Given that $Y \hookrightarrow X$ is a $(\delta, 1)$-equivalence, for δ small, the main result of this section establishes a local homology condition which implies that the inclusion $Y \hookrightarrow X$ is (ε, n)-connected, where ε is a function of δ and n which can be chosen small if n is held fixed and δ is small. While a similar result was established in Quinn [7], it appears that our homology condition has a more local character.

Before stating our main result we will have to introduce some more notation. For any ANR X let $\tilde{X} \xrightarrow{q} X$ be the covering space for which $q^{-1}(C) \xrightarrow{q} C$ is the universal covering space of C, for each component C of X. Whenever convenient we will alternately denote $q^{-1}(A)$ by $\tilde{A}$ and $\tilde{X}|A$, for each A in X. Now choose a collection of basepoints $\{c\} \subset X$, one for each component of X, and define $\pi_1(X, \{c\})$ to be the product group, $\Pi\{\pi_1(X, c) \,|\, c \in \{c\}\}$. For each $c \in \{c\}$ choose $e_c \in q^{-1}(c)$. Then the choice of $\{e_c\}$ determines an isomorphism, $\pi_1(X, \{c\}) \approx G_X$, where G_X is the group of covering transformations of $\tilde{X}$. The singular homology group $H_n(\tilde{X})$ is naturally a ZG_X-module, and it is thus a $Z\pi_1(X, \{c\})$-module once $\{e_c\}$ has been chosen. For any $A \subset X$ note that $H_n(\tilde{X}|A)$ and $H_n(\tilde{X}, \tilde{X}|A)$ are also ZG_X-modules.

Here is the condition needed in our main result below. We say that the pair (X, Y) is *homologically* (ε, δ, n)-*trivial*, for $\delta \leqslant \varepsilon$, if for every $b \in B$ the inclusion-induced homomorphism

$$H_n\big(p^{-1}(S_\delta(b))\big)^{\tilde{}}, \big(p^{-1}(S_\delta(b)) \cap Y\big)^{\tilde{}}\big) \to H_n\big(p^{-1}(S_\varepsilon(b))\big)^{\tilde{}}, \big(p^{-1}(S_\varepsilon(b)) \cap Y\big)^{\tilde{}}\big)$$

is 0. To explain what this homomorphism means we first choose basepoints $\{c\}$ in the components of $p^{-1}(S_\delta(b))$ and $\{e_c\}$ over $\{c\}$ so that the left-hand side of the above map is a $Z\pi_1(p^{-1}(S_\delta(b)), \{c\})$-module. Similarly choose basepoints $\{d\} \subset p^{-1}(S_\varepsilon(b))$ so that $\{c\} \subset \{d\}$ and choose $\{f_d\}$ over $\{d\}$. Then the inclusion

$$i : \big(p^{-1}(S_\delta(b)), p^{-1}(S_\delta(b)) \cap Y\big) \to \big(p^{-1}(S_\varepsilon(b)), p^{-1}(S_\varepsilon(b)) \cap Y\big)$$

is uniquely covered by a map

$$\tilde{i} : \big(p^{-1}(S_\delta(b))\big)^{\tilde{}}, \big(p^{-1}(S_\delta(b)) \cap Y\big)^{\tilde{}}\big) \to \big(p^{-1}(S_\varepsilon(b))\big)^{\tilde{}}, \big(p^{-1}(S_\varepsilon(b)) \cap Y\big)^{\tilde{}}\big)$$

for which e_c is taken to f_c, for all $c \in \{c\}$. This map determines a homomorphism on homology, which is our desired homomorphism. It is important to observe that if

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[2] This is the situation encountered in §6.

this homomorphism is 0 for any one choice of $\{c\}$, $\{e_c\}$, $\{d\}$ and $\{f_d\}$, then it is 0 for any other. Here is our main result.

THEOREM 3.1. *Let an integer $n \geqslant 2$ and a compact metric space B be given. For every $\varepsilon > 0$ there exists a sequence $\{\delta_i\}_{i=1}^{2n}$ of positive numbers such that if (X, Y) and $p: X \to B$ are chosen so that (X, Y) is homologically $(\delta_{2i}, \delta_{2i-1}, i)$-trivial, for $1 \leqslant i \leqslant n$, and $Y \hookrightarrow X$ is a $(\delta_1, 1)$-equivalence, then (X, Y) is (ε, n)-connected.*

As in the case of the Controlled h-Cobordism Theorem there is an addendum which further explains how the δ_i are chosen.

ADDENDUM. *The set $\{\delta_i\}$ can be rechosen as follows: For a fixed i_0 and any $x \in (0, \delta_{i_0})$ we can find another set $\{\delta_i'\}_{i=1}^{2n}$, fulfilling the above requirements, which is of the form $\delta_i' = \delta_i$, for $i > i_0$, and $\delta_{i_0}' = x$.*

We now begin the proof of Theorem 3.1. We will assume that (X, Y) is a compact relative CW pair, i.e., X is obtained from Y by attaching a finite number of cells. After this special case has been taken care of we will show how the argument can be modified to obtain the more general case. The following result gives the basic geometric connection between homology and homotopy that will be needed in our argument. For a proof see [**5**, §8].

LEMMA 3.2. *Assume that X is obtained from Y by attaching cells of dimensions $\geqslant n$ $(n \geqslant 2)$, where 2-cells are trivially attached. If $\{c\} \subset Y$ is a collection of basepoints and $\{e_c\} \subset \tilde{X}$ are chosen over $\{c\}$, then there is a $Z\pi_1(X, \{c\})$-module isomorphism*

$$\pi_n(X, Y, \{c\}) = \sum \left\{ \pi_n(X, Y, c) \mid c \in \{c\} \right\} \approx H_n(\tilde{X}, \tilde{Y})$$

which is uniquely determined by the choice of $\{e_c\}$.

COROLLARY 3.3. *Given the above data assume that each cell of X lies in some $p^{-1}(S_\delta(b))$ and assume that (X, Y) is homologically (ε, δ, n)-trivial. Then for each n-cell of X in $p^{-1}(S_\delta(b))$ there is a homotopy of this n-cell rel ∂ into Y, with this homotopy taking place in $p^{-1}(S_{\varepsilon+2\delta}(b))$.*

PROOF. Let $\phi: (I^n, \partial I^n) \to (p^{-1}(S_\delta(b)), p^{-1}(S_\delta(b)) \cap Y)$ be a characteristic map for the given n-cell of X. Define Z_0 to be $p^{-1}(S_{\varepsilon+2\delta}(b)) \cap Y$ and let Z be the union of Z_0 and all cells of X which meet $p^{-1}(S_\varepsilon(b))$. It then follows that $Z \cap Y = Z_0$. We may regard the homotopy class $[\phi]$ as an element of $\pi_n(p^{-1}(S_\delta(b)), p^{-1}(S_\delta(b)) \cap Y)$, and all we have to do is show that its inclusion-induced image in $\pi_n(Z, Z_0)$ is 0. (For simplicity we have suppressed any mention of basepoints in the various components.) By the idea of Lemma 3.2 above we have an isomorphism $\pi_n(Z, Z_0) \approx H_n(\tilde{Z}, \tilde{Z_0})$, and by the same argument we have a homomorphism of

$$\pi_n\left(p^{-1}(S_\delta(b)), p^{-1}(S_\delta(b)) \cap Y \right)$$

to $H_n(p^{-1}(S_\delta(b))^{\tilde{\,}}, (p^{-1}(S_\delta(b)) \cap Y)^{\tilde{\,}})$ for which the following diagram commutes:

$$
\begin{array}{ccc}
\pi_n\left(p^{-1}(S_\delta(b)), p^{-1}(S_\delta(b)) \cap Y \right) & \to & \pi_n(Z, Z_0) \\
\downarrow & & \downarrow \approx \\
H_n\left(p^{-1}(S_\delta(b))^{\tilde{\,}}, \left(p^{-1}(S_\delta(b)) \cap Y \right)^{\tilde{\,}} \right) & \to & H_n(\tilde{Z}, \tilde{Z_0})
\end{array}
$$

The bottom arrow is inclusion-induced, and it must be 0 since (X, Y) is homologically (ε, δ, n)-trivial. Thus the image of $[\phi]$ in $\pi_n(Z, Z_0)$ is 0. $\square$

Our proof of Theorem 3.1 uses some observations on cell-trading which are explained in the statements below. For notation let $\phi: (I^n, \partial I^n) \to (X, Y)$ be a characteristic map for a cell of X for which there exists a homotopy $\phi \simeq \psi$ rel ∂I^n such that $\psi(I^n) \subset Y$. As explained in [5, §7] the homotopy $\phi \simeq \psi$ can be used to trade this n-cell for a $(n + 2)$-cell, thereby obtaining a compact relative CW pair (X_1, Y) and a homotopy equivalence $g: X \to X_1$ rel Y which preserves skeleta. The following lemma is merely a result of keeping track of how much stretching is involved when the homotopy $\phi \simeq \psi$ has specified control in B.

LEMMA 3.4. *For every $\gamma > 0$ there exists a $\mu > 0$ so that if the homotopy $\phi \simeq \psi$ is a $p^{-1}(\mu)$-homotopy, then there is a γ-equivalence $f: X_1 \to X$ rel Y with γ-inverse g as mentioned above. Moreover there is an integer k (depending on none of the above data) for which $\gamma = k\mu$.*

COROLLARY 3.5. *Given the γ-equivalence above let $p_1 = pf: X_1 \to B$ and let $g: X \to X_1$ be the given γ-inverse of f. It follows that*

(1) *if (X, Y) is homologically (ε, δ, n)-trivial, then (X_1, Y) is homologically $(\varepsilon + 2\gamma, \delta, n)$-trivial,*

(2) *if $A \subset X$ and $\operatorname{diam} p(A) < \varepsilon$, then $\operatorname{diam} p_1 g(A) < \varepsilon + 2\gamma$.*

PROOF. (1) For any $b \in B$ observe that the inclusion $p_1^{-1}(S_\delta(b)) \hookrightarrow p_1^{-1}(S_{\varepsilon+2\gamma}(b))$ factors (up to homotopy) into

$$p_1^{-1}(S_\delta(b)) \xrightarrow{f} p^{-1}(S_\delta(b)) \hookrightarrow p^{-1}(S_\varepsilon(b)) \xrightarrow{g} p_1^{-1}(S_{\varepsilon+2\gamma}(b)).$$

(2) For any $x_1, x_2 \in X$ observe that

$$d(p_1 g(x_1), p_1 g(x_2))$$
$$\leqslant d(p_1 g(x_1), p(x_1)) + d(p(x_1), p(x_2)) + d(p(x_2), p_1 g(x_2)).   \square$$

PROOF OF THEOREM 3.1. It will suffice to consider the case $n = 2$. First assume that the cells of X have been subdivided sufficiently so that they are *small*, i.e., their images in B have small diameters. We will assume that numbers $\delta_1 < \delta_2 < \delta_3 < \delta_4$ are given so that (X, Y) is homologically $(\delta_4, \delta_3, 2)$-trivial and $Y \hookrightarrow X$ is a $(\delta_1, 1)$-equivalence. (The condition that (X, Y) is homologically $(\delta_2, \delta_1, 1)$-trivial is redundant.) We will show that if these numbers are appropriately chosen, then (X, Y) must be $(\varepsilon, 2)$-connected.

Using the fact that $Y \hookrightarrow X$ is a $(\delta_1, 1)$-equivalence we can trade the 0-cells for 2-cells and the 1-cells for 3-cells. By Lemma 3.4 this gives us a relative CW complex (X_1, Y) and a γ_1-equivalence $f_1: X_1 \to X$, where γ_1 is small provided that δ_1 is correspondingly small. Now $p_1 = pf: X_1 \to B$ is our controlling map and it follows from Corollary 3.5 that the cells of X_1 are small (say diameter $\ll \delta_3$ in B) and (X_1, Y) is homologically $(\delta_4 + 2\gamma_1, \delta_3, 2)$-trivial. Since $f_1: X_1 \to X$ is a γ_1-equivalence and $Y \hookrightarrow X$ is a $(\delta_1, 1)$-equivalence, it follows that $Y \hookrightarrow X_1$ is a $(\delta_1 + \gamma_1, 1)$-equivalence. Thus we may assume that each 2-cell of X_1 lies in some $p_1^{-1}(S_{\delta_3}(b))$, and the attaching map for this 2-cell is null-homotopic in $p_1^{-1}(S_{\delta_3}(b)) \cap Y$. This enables us to use the

idea of the proof of Corollary 3.3 to obtain a homotopy of each such 2-cell into Y, where this homotopy takes place in some $p^{-1}(S_{\delta_4 + 2\gamma_1 + 2\delta_3}(b))$. Thus we can trade the 2-cells for 4-cells to obtain a relative CW complex (X_2, Y) with cells of dimension $\geqslant 3$ and a $p_1^{-1}(\gamma_2)$-equivalence $f_2: X_2 \to X_1$. Clearly γ_2 is small if $\delta_4 + 2\gamma_1 + 2\delta_3$ is small.

To see that (X, Y) is $(\varepsilon, 2)$-connected let $\phi: (I^2, \partial I^2) \to (X, Y)$ be a map. Choose the subdivision of X and adjust ϕ so that $\phi(I^2)$ lies in the 2-skeleton of X, say $\phi(I^2) \subset X^2$. We may choose $g_1: X \to X_1$ to be a γ_1-inverse of f_1 which is skeleta-preserving. Similarly let $g_2: X_1 \to X_2$ be chosen to preserve skeleta, but observe that we now have $X_2^2 = Y$. Thus $g_2 g_1 \phi(I^2) \subset Y$, and we have a $p^{-1}(\varepsilon)$-homotopy

$$\phi \simeq f_1 f_2 g_2 g_1 \phi \text{ rel } \partial I^2. \quad \square$$

REMARKS. To see how the above argument adapts to the case in which X is an ANR which is ε-finitely dominated for all $\varepsilon > 0$, we first have to replace X by a (possibly noncompact) relative CW complex. This is accomplished by using the mapping telescope construction with control, as explained in [2]. It asserts that for every $\gamma_1 > 0$ we can choose a relative CW complex (X_1, Y) and a γ_1-equivalence $f_1: X_1 \to X$ rel Y. Let $p_1 = pf: X_1 \to B$ be a controlling map for X_1, and it follows that X_1 is ε_1-finitely dominated, where ε_1 is a small number provided that γ_1 is small. By Proposition 5.1 (see Remark 1) we can attach a finite number of cells to Y to obtain a finite relative CW complex X_2 and a map $f_2: X_2 \to X_1$ which is a γ_2-domination rel Y and which is $(\gamma_2, n + 2)$-connected (with respect to the controlling map $p_1: X_1 \to B$). Of course the size of γ_2 depends on the size of ε_1. If we let $X' = X_2$ and $f' = f_1 f_2$, then it is easy to see that f' is $(\gamma, n + 2)$-connected, where the size of γ depends on the sizes of γ_1 and γ_2.

Let $p' = pf': X' \to B$ be a controlling map for X' and assume that (X, Y) is homologically $(\delta_{2i}, \delta_{2i-1}, i)$-trivial and $Y \hookrightarrow X$ is a $(\delta_1, 1)$-equivalence. Again it is easy to see that $Y \hookrightarrow X'$ is a $(\delta_1', 1)$-equivalence, where the size of δ_1' depends on the sizes of γ and δ_1. It is our intention to show that (X', Y) is homologically $(\delta_{2i}', \delta_{2i-1}, i)$-trivial, where the size of δ_{2i}' depends on the sizes of δ_{2i} and γ. Once this has been done the above argument for the compact relative CW case applies to give (X', Y) (ε', n)-connected, and this in turn implies that (X, Y) is (ε, n)-connected. We can subdivide the open set $(p')^{-1}(S_{\delta_{2i}}(b))$ of X' so that it is a relative CW complex. If $(p')^{-1}(S_{\delta_{2i}}(b))^k$ is its k-skeleton, then we only have to show that the following homomorphism is 0:

$$H_i\left(\left((p')^{-1}\left(S_{\delta_{2i-1}}(b)\right)^{n+1}\right)^{\tilde{}}, \left((p')^{-1}\left(S_{\delta_{2i}}(b)\right) \cap Y\right)^{\tilde{}}\right)$$

$$\to H_i\left(\left((p')^{-1}\left(S_{\delta_{2i}'}(b)\right)\right)^{\tilde{}}, \left((p')^{-1}\left(S_{\delta_{2i}'}(b)\right) \cap Y\right)^{\tilde{}}\right).$$

First observe that (X_1, Y) is homologically $(\delta_{2i + \gamma_1}, \delta_{2i-1}, i)$-trivial. Indeed the inclusion $(pf_1)^{-1}(S_{\delta_{2i-1}}(b)) \hookrightarrow (pf_1)^{-1}(S_{\delta_{2i} + \gamma_1}(b))$ factors (up to homotopy) into

$$\left(pf_1\right)^{-1}\left(S_{\delta_{2i-1}}(b)\right) \xrightarrow{f_1} p^{-1}\left(S_{\delta_{2i-1}}(b)\right) \hookrightarrow p^{-1}\left(S_{\delta_{2i}}(b)\right) \xrightarrow{g_1} \left(pf_1\right)^{-1}\left(S_{\delta_{2i} + \gamma_1}(b)\right),$$

where g_1 is a γ_1-inverse of f_1. By construction the map $f_2: X_2 \to X_1$ is a γ_2-domination with inverse $g_2: X_1 \to X_2$ for which the inclusion

$$\left(p'\right)^{-1}\left(S_{\delta_{2i-1}}(b)\right)^{n+1} \hookrightarrow \left(p'\right)^{-1}\left(S_{\delta_{2i}'}(b)\right)$$

is homotopic to the composition

$$(p')^{-1}\big(S_{\delta_{2i-1}}(b)\big)^{n+1} \xrightarrow{f_2} (pf_1)^{-1}\big(S_{\delta_{2i}}(b)\big) \rightsquigarrow (pf_1)^{-1}\big(S_{\delta_{2i}+\gamma_1}(b)\big)$$

$$\xrightarrow{g_2} (p')^{-1}\big(S_{\delta_{2i}+\gamma_1+\gamma_2}(b)\big)$$

(see Remark 2 following Proposition 5.1). This is all we need. $\square$

4. Controlled matrices. The purpose of this section is to formulate the algebraic machinery which is needed for the proof of the Controlled Boundary Theorem that is given in §6. Here is a brief description of our problem. Let (K, L) be a compact polyhedral pair, where K is formed by attaching k n-cells to L so that all have the same basepoint, i.e., the ith one has attaching map $\phi_i : (\partial I^n, *) \to (L, l_0)$, for l_0 a fixed point of L. It is well known that for $n \geqslant 2$ (where the cells are trivially attached in case $n = 2$) the $Z\pi_1(L)$-module $\pi_n(K, L, l_0)$ is isomorphic to the direct sum of k-copies of $Z\pi_1(L)$. In this section we will treat a similar situation in which the n-cells of K have different basepoints, and our problem will be to describe homotopy classes of maps $(I^n, \partial I^n) \to (K, L)$. To compound the problem these homotopy classes will be controlled over our parameter space B. In order to capture this full generality we have devised a notion of controlled matrices. The building blocks for these controlled matrices are the path groups as defined below.

First we treat the uncontrolled situation. For any space X and $x, y \in X$ we define the *path group* $P[x, y]$ to be the free abelian group generated by the set of homotopy classes of paths from x to y. If no such paths exist then we define $P[x, y] = 0$. Thus $P[x, y]$ consists of all finite formal sums $\sum m_r[w_r]$, where w_r is a path from x to y. Now let $\hat{X} = X \cup e_1^n \cup \cdots \cup e_k^n$, $n \geqslant 2$, where e_j^n is an n-cell attached to X with attaching map $\phi_j : (\partial I^n, *) \to (X, y_j)$. Observe that for each $x \in X$ there exists a natural homomorphism $\alpha_j : P[x, y_j] \to \pi_n(\hat{X}, X, x)$ defined by sending $\sum m_r[w_r]$ to $\sum m_r[w_r^{-1}\hat{\phi}_j]$, where $\hat{\phi}_j : (I^n, \partial I^n, *) \to (\hat{X}, X, y_j)$ is a characteristic map for e_j^n whose restriction to ∂I^n is ϕ_j and $w_r^{-1}\hat{\phi}_j$ changes the basepoint by dragging the basepoint of e_j^n along the path w_r^{-1} from y_j to x. Direct summing gives us a homomorphism

$$\alpha = \sum \alpha_j : \sum P[x, y_j] \to \pi_n(\hat{X}, X, x).$$

We can easily give $\sum P[x, y_j]$ a $Z\pi_1(X, x)$-module structure so that α is a $Z\pi_1(X, x)$-module isomorphism. Then it easily follows from [5, §8] that α is an isomorphism provided that the e_j^n are trivially attached if $n = 2$ (cf. Lemma 3.2 of this paper).

The above comments describe the classical situation, and we now introduce notation to describe the controlled case. Continuing the above notation let $p : X \to B$ be a controlling map, let $N_\varepsilon(x) = p^{-1}(S_\varepsilon(p(x)))$, and let

$$\hat{N}_\varepsilon(x) = \cup \big\{ e_j^n \mid \phi_j(\partial I^n) \subset N_\varepsilon(x) \big\} \cup N_\varepsilon(x).$$

We then define $P_\varepsilon[x, y]$ to be the ε-*path* group of all $\sum m_r[w_r]$, where w_r is a path in $N_\varepsilon(x)$ from x to y. As usual, $P_\varepsilon[x, y] = 0$ if there are no such paths. In what follows

we assume that $\operatorname{diam}(p\phi_j(\partial I^n)) < \varepsilon_0$, for all j. Then for every $\varepsilon > 0$ there exists a natural homomorphism

$$\alpha_\varepsilon : \sum P_\varepsilon[x, y_j] \to \pi_n\big(\hat{N}_{\varepsilon+\varepsilon_0}(x), N_{\varepsilon+\varepsilon_0}(x), x\big)$$

which is defined in analogy with the homomorphism α described above. It is clear that the following diagram commutes for $\varepsilon \leqslant \delta$:

$$
\begin{array}{ccc}
\sum P_\varepsilon[x, y_j] & \overset{\alpha_\varepsilon}{\to} & \pi_n\big(\hat{N}_{\varepsilon+\varepsilon_0}(x), N_{\varepsilon+\varepsilon_0}(x), x\big) \\
\downarrow & & \downarrow \\
\sum P_\delta[x, y_j] & \overset{\alpha_\delta}{\to} & \pi_n\big(\hat{N}_{\delta+\varepsilon_0}(x), N_{\delta+\varepsilon_0}(x), x\big)
\end{array}
$$

If β is the inverse of the isomorphism α, then for every $\varepsilon > 0$ there is an ε-version of this homomorphism,

$$\beta_\varepsilon : \pi_n\big(\hat{N}_\varepsilon(x), N_\varepsilon(x), x\big) \to \sum P_\varepsilon[x, y_j].$$

This satisfies the property that the following diagram commutes for $\varepsilon \leqslant \delta$:

$$
\begin{array}{ccc}
\pi_n\big(\hat{N}_\varepsilon(x), N_\varepsilon(x), x\big) & \overset{\beta_\varepsilon}{\to} & \sum P_\varepsilon[x, y_j] \\
\downarrow & & \downarrow \\
\pi_n\big(\hat{N}_\delta(x), N_\delta(x), x\big) & \overset{\beta_\delta}{\to} & \sum P_\delta[x, y_j]
\end{array}
$$

Just as it was observed that α is an isomorphism, it similarly follows that $\beta_{\varepsilon+\varepsilon_0}\alpha_\varepsilon$ and $\alpha_\varepsilon\beta_\varepsilon$ are inclusion-induced, for all $\varepsilon > 0$.

By a *path matrix* $A = (a_{ij})$ on X we mean a finite matrix whose ith row corresponds to $x_i \in X$ and whose jth column corresponds to $y_j \in X$ (with possible repetitions), and $a_{ij} \in P[x_i, y_j]$. An *ε-path matrix* is essentially the same thing except that $a_{ij} \in P_\varepsilon[x_i, y_j]$. The *domain* of A is the ordered tuple (x_i), and the *range* of A is the ordered tuple (y_j), written $A : (x_i) \to (y_j)$. If $A = (a_{ij}) : (x_i) \to (y_j)$ is an ε-path matrix and $B = (b_{jk}) : (y_j) \to (z_k)$ is a δ-path matrix, then $AB : (x_i) \to (z_k)$ is the $(\varepsilon + \delta)$-path matrix $(\sum_j a_{ij} * b_{jk})$. Here the symbol $*$ is used to denote the composition of homotopy classes of paths.

In the sequel we will use ε-path matrices to describe controlled homotopy classes as was discussed earlier. The following remarks are intended to make it clear how ε-path matrices will be used. Just as above $p : X \to B$ will be a controlling map and $(\hat{X}, X)$ will be a relative CW complex so that $\operatorname{diam}(p\phi_j(\partial I^n)) < \varepsilon_0$, where $\phi_j : (\partial I^n, *) \to (X, y_j)$ is the attaching map for e_j^n. Let (x_i) be any finite ordered tuple of points of X.

1. If $A : (x_i) \to (y_j)$ is an ε-path matrix, then for each i the ith row of A determines a map $\psi_i : (I^n, \partial I^n, *) \to (\hat{X}, X, x_i)$ so that $\psi_i(I^n) \subset \hat{N}_{\varepsilon+\varepsilon_0}(x_i)$. ψ_i arises from the homomorphism α_ε.

2. If $\psi_i : (I^n, \partial I^n, *) \to (\hat{X}, X, x_i)$ is a map so that $\psi_i(I^n) \subset \hat{N}_\varepsilon(x_i)$, then there is an ε-path matrix $A : (x_i) \to (y_j)$ whose ith row is determined by ψ_i. The elements of this row arise from the homomorphism β_ε.

5. Controlled Wall theory. The purpose of this section is to establish controlled versions of some aspects of Wall's finiteness obstruction [**10**]. As in §4 we will have to deal with the problem of multiple basepoints, so we will be using the controlled matrices which were defined there. We will first establish the following result which treats the notion of controlled cellular surgery on a map. For notation let K and L be polyhedra such that K is compact and let $p: L \to B$ be a controlling map to our parameter space such that L is ε-finitely dominated, for all $\varepsilon > 0$.

PROPOSITION 5.1. *For every $\varepsilon > 0$ and $n \geq 0$ there exists a $\delta > 0$ so that if $f: K \to L$ is $(\delta, n - 1)$-connected, then by attaching a finite number of n-cells to K we can form $\tilde{K}$ and extend f to a map $\tilde{f}: \tilde{K} \to L$ which is (ε, n)-connected. Moreover, δ depends only on ε and n.*

PROOF. By the standard mapping cylinder trick we may assume that K is a subpolyhedron of L and f is the inclusion $K \hookrightarrow L$. Choose a fine subdivisiion of L and let L^k be its k-skeleton. By using the $(\delta, n - 1)$-connectivity of $K \hookrightarrow L$ we can inductively work through the skeleta of L^{n-1} to construct a map $\phi: K \cup L^{n-1} \to K$ for which $\phi | K = \text{id}$ and ϕ is $p^{-1}(\delta')$-homotopic to $K \cup L^{n-1} \hookrightarrow L$ rel K, where δ' is small if δ is small. Since L is ε-finitely dominated, for all $\varepsilon > 0$, we can find a compact subpolyhedron J of L containing K for which there is a $p^{-1}(\delta)$-homotopy of L into J rel K. In the following construction of $\tilde{f}: \tilde{K} \to L$ it will be clear that if δ and δ' are small enough, then the desired ε-control on the n-connectivity of $\tilde{f}$ is achieved.

We define $\tilde{K}$ to be the quotient space $(K \cup J^n)/\sim$, where $\sim$ is the equivalence relation generated by $x \sim \phi(x)$, for all $x \in J^{n-1}$. Let $q: K \cup J^n \to \tilde{K}$ be the resulting quotient map. To define $\tilde{f}: \tilde{K} \to L$ choose any n-simplex $\Delta^n \subset J^n$ and note that $q(\Delta^n)$ is a typical n-cell in $\tilde{K}$ which is attached to K. Since we want $\tilde{f} | K = \text{id}$ we will only have to concentrate on the definition of $\tilde{f}$ on $q(\Delta^n)$. For this it will be convenient to define $\Delta^n = [-1, 1]^n$ and, more generally, $\Delta^n_t = [-t, t]^n$. On $q(\Delta^n_{1/2})$ we define $\tilde{f}$ to be γq^{-1}, where $\gamma: \Delta^n_{1/2} \to \Delta^n$ is a radially-defined homeomorphism. On $q(\partial \Delta^n_t)$, $\frac{1}{2} \leq t \leq 1$, we define $\tilde{f}$ to be $\gamma_t q^{-1}$ followed by the $(2t - 1)$-level of the homotopy $\text{id} \simeq \phi$, where $\gamma_t: \partial \Delta^n_t \to \partial \Delta^n$ is a radially-defined homeomorphism.

Finally we have to show that $\tilde{f}: \tilde{K} \to L$ is (ε, n)-connected. So let $g: I^k \to L$, $g_0: \partial I^k \to \tilde{K}$ be maps for which $\tilde{f}g_0 = g | \partial I^k$, for $k \leq n$. By general positioning we may assume that $g_0(\partial I^k) \subset K$. We want a map $\tilde{g}: I^k \to \tilde{K}$ for which $\tilde{g} | \partial I^k = g_0$ and $\tilde{f}\tilde{g} \simeq g$ rel ∂I^k via a $p^{-1}(\varepsilon)$-homotopy. To do this we first deform g so that it is into J, and then by general positioning we may assume that it is into J^n. Finally composing with $q: K \cup J^n \to \tilde{K}$ gives us our desired map $\tilde{g}: I^k \to \tilde{K}$. It is easy to see that $\tilde{f}\tilde{g} \simeq g$ rel ∂I^k via a $p^{-1}(\varepsilon)$-homotopy (for appropriately small choices of δ and δ'). $\square$

REMARKS. 1. The above proof easily applies to the more general setting as encountered in the proof of Theorem 3.1. Suppose we are given a relative CW complex (L, K), K compact, and a map $p: L \to B$. For every $\varepsilon > 0$ and $n \geq 0$ there exists a $\delta > 0$ (depending only on ε and n) so that if L is δ-finitely dominated, then there exists a compact relative CW complex $(\tilde{K}, K)$ and a map $f: \tilde{K} \to L$ which is (ε, n)-connected and which is an ε-domination rel K. To prove this let $K_0 \supset K$ be a

compact subcomplex of L for which $K_0 \hookrightarrow L$ is a δ-domination rel K. Thus $K_0 \hookrightarrow L$ is $(\delta, 0)$-connected, so we can apply the above proof n times to the inclusion $K_0 \hookrightarrow L$ to extend it to our desired $f: \tilde{K} \to L$.

2. Also in the above general setting it follows that if (J^{n-1}, K) is a relative CW complex and $\phi: J^{n-1} \to \tilde{K}$ is a map such that $\phi|K = \mathrm{id}$, then there exists a $(pf)^{-1}(\varepsilon_1)$-homotopy $\phi \simeq gf\phi$ rel K, where $g: L \to \tilde{K}$ is a right ε-inverse of f and ε_1 is small if ε is small. This is done by using the (ε, n)-connectivity of f to lift the $p^{-1}(\varepsilon)$-homotopy $f\phi \simeq (fg)f\phi$ from L to $\tilde{K}$.

We now establish a result which says that in certain situations cells can be traded down to lower-dimensional cells with control in the parameter space B. For notation let (J, K) and (L, J) be compact relative CW complexes, where $J = K \cup \{(n + 1)\text{-}$ cells$\}$ and $L = J \cup \{(n + 2)\text{-cell}\}$, $n \geq 2$. Also let $p: L \to B$ be a controlling map to our parameter space.

LEMMA 5.2. *For every $\varepsilon > 0$ there exists a $\delta > 0$ so that if $J \hookrightarrow L$ is a δ-domination rel K and the images of the $(n + 1)$- and $(n + 2)$-cells of L have diameters $< \delta$ in B, then there exist compact relative CW complexes (J', K) and (L', J'), where $J' = K' \cup \{n\text{-cells}\}$ and $L' = J' \cup \{(n + 1)\text{-cells}\}$, and an ε-equivalence $L' \simeq L$ rel K. Moreover, δ depends only on ε.*

PROOF. Let the $(n + 1)$-cells of J have characteristic maps $\phi_i: (I^{n+1}, \partial I^{n+1}, *) \to (J, K, x_i)$, $1 \leq i \leq k$, and let the $(n + 2)$-cells of L have characteristic maps $\psi_j: (I^{n+2}, \partial I^{n+2}, *) \to (L, J, y_j)$, $1 \leq j \leq l$. We may choose $y_j \in K$ since the cells have small diameters in B. The definition of J' is easy. It is simply formed by attaching l n-cells to K, where the jth one is trivially attached at $y_j \in K$. The definition of L' is a little harder. There will be k $(n + 1)$-cells of L'—one for each $(n + 1)$-cell of J. The ith $(n + 1)$-cell will be based at x_i, and the attaching map for this cell will be a map $\theta_i: (I^{n+1}, *) \to (J', x_i)$ as described below.

Our hypotheses give us a map $g: L \to J$ which satisfies $g \simeq \mathrm{id}$ rel K via a $p^{-1}(\delta)$-homotopy. So for each i the homotopy exact sequence of (L, J, K) (based at x_i) gives us a split short exact sequence,

$$0 \to \pi_{n+2}(L, J) \underset{r_i}{\to} \pi_{n+1}(J, K) \underset{u_i}{\to} \pi_{n+1}(L, K) \to 0.$$

The splitting map u_i is induced by g, and the splitting map r_i enjoys the property that $r_i([\phi_i]) = [\tilde{\phi}_i]$, where $\tilde{\phi}_i: (I^{n+2}, \partial I^{n+2}, *) \to (L, J, x_i)$ is a map for which diam $p\tilde{\phi}_i(I^{n+2})$ is small provided that δ is small. The entire collection of maps $\{\tilde{\phi}_i\}$ determines a δ'-path matrix $R: (x_i) \to (y_j)$. The ith row of R determines a map $\theta_i': (\partial I^{n+1}, *) \to (J', x_i)$ (since the cells of J' are trivially attached). We also have a map $\phi_i|\partial I^{n+1}: (\partial I^{n+1}, *) \to (K, x_i) \subset (J', x_i)$, and so we can form a map $\theta_i: (\partial I^{n+1}, *) \to (J', x_i)$ for which $[\theta_i] = [\theta_i'] + [\phi_i|\partial I^{n+1}]$. This map is used as an attaching map to attach the ith $(n + 1)$-cell of L' to J', thus completing the definition of L'. Observe that δ' can be chosen small provided that δ is also chosen small. In what follows below it will be convenient to write $J = K \cup d_1 \cup \cdots \cup d_k$ and $L = J \cup e_1 \cup \cdots \cup e_l$, where d_i has characteristic map ϕ_i and e_j has characteristic map ψ_j. Similarly let $J' = K \cup e_1' \cup \cdots \cup e_l'$ and $L' = J' \cup d_1' \cup \cdots \cup d_k'$.

Our method of showing that L' is homotopy equivalent to L rel K is to trade the n-cells of J' for $(n + 2)$-cells, thus forming CW complexes

$$J'' = K \cup \{(n + 1)\text{-cells}\} = K \cup d_1'' \cup \cdots \cup d_k'',$$

$$L'' = J'' \cup \{(n + 2)\text{-cells}\} = J'' \cup e_1'' \cup \cdots \cup e_l''.$$

In order that L'' be homotopy equivalent to L with ε-control in B it suffices that the following be true (cf. [5, 7.1]):

(1) the attaching maps of d_i and d_i'' are homotopic (small homotopy in B),

(2) the attaching maps of e_j and e_j'' are homotopic (small homotopy in B).

Our desired controlled homotopy equivalence $L' \simeq L$ is then a composition $L' \simeq L'' \simeq L$, where the first homotopy equivalence easily results from the cell-trading process.

To carry out the cell-trading process we will have to obtain a homotopy of each e_j' into K, with the homotopy taking place in L'. This is done by finding an element of $\pi_{n+1}(L', J', y_j)$ whose image in $\pi_n(J', K, y_j)$ under the boundary homomorphism $\partial_j' : \pi_{n+1}(L', J', y_j) \to \pi_n(J', K, y_j)$ is the homotopy class of e_j. This element of $\pi_{n+1}(L', J')$ then provides a homotopy of e_j' into K as desired. Of course all of this must be done with small control in B. Our first step is to obtain a path matrix representing the boundary homomorphisms

$$\partial_j : \pi_{n+2}(L, J, y_j) \to \pi_{n+1}(J, K, y_j).$$

For each j the restriction $\psi_j| : (\partial I^{n+2}, *) \to (J, y_j)$ may be regarded as a map of $(I^{n+1}, \partial I^{n+1}, *)$ into (J, K, y_j), and so the entire collection of maps determines a 2δ-path matrix $\partial : (y_j) \to (x_i)$. The next step is to interpret the jth row of ∂ as a map $\sigma_j : (I^{n+1}, \partial I^{n+1}, *) \to (L', J', y_j)$, thus given $[\sigma_j] \in \pi_{n+1}(L', J', y_j)$. Now observe that $\partial_j'([\sigma_j])$ is determined by the jth row of the product matrix ∂R, and this is the identity matrix by the manner in which the r_i were chosen. This means that $\partial_j'([\sigma_j])$ is the homotopy class of e_j', and so σ_j can be used to deform e_j' into K. Therefore the cell-trading process can be carried out to trade the e_j' for $(n + 2)$-cells e_j''. At the same time the d_i' are deformed to new $(n + 1)$-cells d_i''.

Since the e_j' are trivially attached the R-component of the attaching map of a typical d_i' disappears when it is deformed to d_i'', and so d_i'' is attached to K according to the map $\phi_i | \partial I^{n+1}$. Also the homotopy used to deform a typical e_j' into K becomes the attaching map of e_j'', and so the e_j'' are attached to J'' according to the matrix ∂. This is in agreement with the situation for J and L. $\square$

We are now ready for the main result of this section. Here is a brief description of our problem. Let (L, J, K) be a polyhedral triple (all connected), where J and K are compact and J is formed by attaching n-cells to K, $n \geqslant 2$. Assume further that (L, J) is n-connected and the inclusion $J \hookrightarrow L$ is a domination rel K. One easily shows that the inclusion-induced homomorphism of $Z\pi_1(K)$-modules, $\pi_n(J, K) \to \pi_n(L, K)$, is split, so $\pi_n(J, K) \approx \pi_{n+1}(L, J) \oplus \pi_n(L, K)$. It follows from the results of [10] that if L is homotopy equivalent to a finite complex, then $\pi_n(L, K)$ is stably free. This quickly implies that there are free f.g. $Z\pi_1(K)$-modules F_1, F_2 and an isomorphism

$$\pi_n(J, K) \oplus F_1 \oplus F_2 \approx \pi_n(J, K) \oplus F_1 \oplus F_2$$

which takes $\pi_n(J, K) \oplus F_1$ onto $\pi_n(L, K) \oplus F_2$ and which takes F_2 onto $\pi_{n+1}(L, J)$ $\oplus F_1$. The next result establishes a controlled version of this situation.

For notation let (J, K) and (L, K) be relative CW complexes, where J is compact and it is formed by attaching n-cells to K, $n \geqslant 2$, with basepoints $x_1, x_2, \ldots, x_r$. Also let $p: L \to B$ be a controlling map to our parameter space and let $f: J \to L$, $f|K = \text{id}$, be a map which is (δ, n)-connected and which is a δ-domination rel K. If $g: L \to J$ is a δ-inverse of f rel K, then we get a map $gf: J \to J$. Let $pf: J \to B$ be a controlling map for J and assume that the diameters of the images of the n-cells of J in B are less than δ. According to the notation of §4 the images of the n-cells of J under the gf yield a 2δ-path matrix $A: (x_i) \to (x_i)$. If $I: (x_i) \to (x_i)$ is the identity path matrix, then we can also form the 2δ-path matrix $I - A: (x_i) \to (x_i)$, where $-A$ is formed by multiplying each element of A by -1 and addition is just addition in the path groups.

THEOREM 5.3. *Let $\varepsilon > 0$ and an integer $m > 0$ be given. The number δ mentioned above can be chosen small enough so that if there exists a compact relative CW complex (P, K) of dimension $\leqslant m$ and a δ-equivalence $P \simeq L$ rel K, then there exists tuples (y_j) and (z_k) on K and an ε-path isomorphism $B: ((x_i), (y_j), (z_k)) \to ((x_i), (y_j), (z_k))$ for which*

$$\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix} B \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix} = 0$$

and

$$\begin{pmatrix} I - A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} B \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

REMARKS. The number δ in the above statement depends only on ε, n and m. Also the rows and columns in the above matrices are arranged in blocks with the order being (x_i), (y_j) and (z_k).

PROOF. By a mapping cylinder construction we may assume that $J \subset P$ so that (P, J) is a relative CW complex. Since we have a δ-equivalence $P \simeq L$ we may assume that $J \hookrightarrow P$ is a δ-domination rel K and (P, J) is (δ, n)-connected, i.e., $J \hookrightarrow P$ is (δ, n)-connected. Thus we can use cell-trading to further assume that $P - J$ has no cells of dimension $< n$ (cf. Lemma 3.4). (Our controlling map $P \to B$ is the composition $P \simeq L \xrightarrow{p} B$.) Let P^i denote the i-skeleton of P, i.e., P^i is the union of J and all cells of $P - J$ of dimension $\leqslant i$. By the controlled homotopy extension theorem [4, §2] we observe that if $i \geqslant n$, then $P^{i+1} \hookrightarrow P$ is a δ'-domination rel P^i, where δ' is small if δ is small. Thus by inductive use of Lemma 5.2 the cells of P can be traded down. This means that we may additionally assume $m = n + 1$. So in this simplified model P consists of J with a finite number of n-cells attached to J, and then a finite number of $(n + 1)$-cells attached to this. Since (P, J) is (δ, n)-connected these additional n-cells of P may be taken to be trivially attached to K. Let J_1

be the union of K and these trivially attached n-cells, say attached at (y_j) in K. Let (z_k) in K be the basepoints for the remaining $(n + 1)$-cells of P. Here is a picture of P.

K	$J_1 \{n\text{-cells}\}$	{$n + 1$}-cells
	$J \{n\text{-cells}\}$	

Now for some notation. If $h : P \to L$ is the given δ-equivalence, then $ph : P \to B$ is our controlling map to B and we may assume that diam $ph(e)$ is small, for each cell e of P. We are given a map $g' : P \to J$ for which $g' \simeq \text{id rel } K$, where this is a $(ph)^{-1}(\delta)$-homotopy. The restriction $g'| : J \to J$ is the map that produces our given matrix A. Since $J \cup J_1 \hookrightarrow P$ is a δ-domination rel K we are in a situation similar to that which arose in the proof of Lemma 5.2. So if $\partial : (z_k) \to ((x_i), (y_j))$ is the 2δ-path matrix determined by the attaching maps for the $(n + 1)$-cells of P, then for δ' small there is a δ'-path matrix $R : ((x_i), (y_j)) \to (z_k)$ for which $\partial R = I_{(z_k)}$. It is also easily seen that $R\partial = (I - A) \oplus I_{(y_j)}$. Now define our desired matrix $B : ((x_i), (y_j), (z_k)) \to ((x_i), (y_j), (z_k))$ by

$$B = \begin{pmatrix} A & 0 & (I - A)R \\ 0 & 0 & R \\ \partial_x & \partial_y & 0 \end{pmatrix}.$$

This has an inverse,

$$B^{-1} = \begin{pmatrix} A & 0 & R \\ 0 & 0 & R \\ \partial_x & \partial_y & 0 \end{pmatrix},$$

so it is an isomorphism. If one observes that $A^2 = A$ and $(I - A)^2 = I - A$, then it is easy to check that B fulfills our requirements. $\square$

6. Proof of the controlled boundary theorem. We are given $p : M^m \to B$ which is tame at ∞, 1-movable at ∞, and nice at ∞. Our strategy for putting a controlled boundary on M is similar to that of Siebenmann [8] or Quinn [7]. The first step is to show that for every sequence $\varepsilon_1 > \varepsilon_2 > \cdots$, $\lim \varepsilon_i = 0$, there are neighborhoods $N_1 \supset\supset N_2 \supset\supset \cdots$ of ∞, $\cap N_i = \varnothing$, so that $\partial N_i \hookrightarrow N_i - \mathring{N}_{i+1}$ and $\partial N_{i+1} \hookrightarrow N_i - \mathring{N}_{i+1}$ are $(\varepsilon_i, 1)$-equivalences. (Notation such as this will be used throughout this section; it means that each N_i is a clean submanifold of M for which $M - \mathring{N}_i$ is compact and $N_{i+1} \subset \mathring{N}_i$.) This first step is just a controlled version of the corresponding step in Siebenmann [8] where small 1-neighborhoods of ∞ are constructed. The next step is to show that the sequence $\{N_i\}$ can be chosen to additionally satisfy the property that $(N_i, \partial N_i)$ is $(\varepsilon_i, m - 3)$-connected, i.e., $\partial N_i \hookrightarrow N_i$ is $(\varepsilon_i, m - 3)$-connected. These two steps only use the tameness at ∞ and the 1-movability at ∞. The third step is to use the niceness at ∞ to rechoose the N_i so that $(N_i, \partial N_i)$ is $(\varepsilon_i, m - 2)$-connected. Finally the last step is to explain why the existence of these

$(m-2)$-neighborhoods implies that M admits a controlled boundary. There are many parts of the argument of Quinn [7] which apply here even though we do not assume that B is locally simply connected and we do not assume that π_1 is locally constant. We shall take advantage of this similarity to significantly simplify some of the steps below.

I. 1-*Neighborhoods of* ∞. Given any $\varepsilon > 0$ we will first show how to construct a neighborhood N of ∞ so that $\partial N \hookrightarrow N$ is an $(\varepsilon, 1)$-equivalence. In order to do this we only need the tameness of p at ∞. Let $N' \subset M$ by any neighborhood of ∞. By repeated application of Proposition 5.1 it follows that for every $\varepsilon > 0$ we can attach a finite number of 0-, 1- and 2-cells to $\partial N'$, thus forming a CW complex K, and extend $\partial N' \hookrightarrow N'$ to a map $f: K \to N'$ which is $(\varepsilon, 2)$-connected. Since $m \geq 5$ we can embed K in N' rel $\partial N'$ and take f to be the inclusion $K \hookrightarrow N'$. Let $R \subset M$ be a regular neighborhood of $\overline{M - N'} \cup K$ and let $N = \overline{M - R}$. If R is sufficiently close to $\overline{M - N'} \cup K$ and $m \geq 5$, then it is easy to check that $\partial N \hookrightarrow N$ is an $(\varepsilon, 1)$-equivalence simply by using the fact that $K \hookrightarrow N'$ is $(\varepsilon, 2)$-connected along with general positioning. Observe that if $m \geq 6$, then this argument shows that $\partial N \hookrightarrow N$ is $(\varepsilon, 2)$-connected.

The next step is to perform a similar maneuver on the left, and for this we only need the 1-movability of p at ∞. Let $C \subset D$ be compacta in M and let $N \subset M - D$ be a neighborhood of ∞. We say that $f: K \to \overline{M - N} - D$ is an (ε, n)-equivalence (or is (ε, n)-connected) in $\overline{M - N} - C$ provided that the lifting property which appears in the related definitions of §2 is satisfied, but the homotopy $f\tilde{\phi} \simeq \phi$ is allowed to take place in $\overline{M - N} - C$. If we are to perform our maneuver on the left as above we can no longer directly apply the statement of Proposition 5.1, however the proof still applies. Using the proof of Proposition 5.1 it follows that for every $\varepsilon > 0$ and compactum $C \subset M$, there exists a compactum $D \supset C$ which satisfies the following property: for every neighborhood $N' \subset M - D$ of ∞ which is sufficiently close to ∞, we can attach a finite number of 0-, 1-, and 2-cells to $\partial N'$ to form K so that the inclusion $\partial N' \hookrightarrow \overline{M - N'} - D$ extends to a map $f: K \to \overline{M - N'} - D$ which is an $(\varepsilon, 1)$-equivalence in $\overline{M - N'} - C$. Since $m \geq 5$ we may take K to be in $\overline{M - N'} - D$ so that f is the inclusion.

Now let N be a regular neighborhood of $N' \cup K$. If N is sufficiently close to $N' \cup K$ it follows that $\partial N \hookrightarrow \overline{M - N} - D$ is an $(\varepsilon, 1)$-equivalence in $\overline{M - N} - C$. Note that if N' additionally satisfied the property that $\partial N' \hookrightarrow N'$ is an $(\varepsilon, 1)$-equivalence, then $K \hookrightarrow N' \cup K$ is still an $(\varepsilon', 1)$-equivalence (by definition), where ε' is small if ε is small. So by a general position argument it follows that $\partial N \hookrightarrow N$ is an $(\varepsilon', 1)$-equivalence. Thus we have shown that for every $\varepsilon > 0$ and compactum $C \subset M$ there exists a compactum $D \supset C$ which satisfies the following property: for every compactum $E \supset D$ there exists a neighborhood $N \subset M - E$ of ∞ such that $\partial N \hookrightarrow N$ is an $(\varepsilon, 1)$-equivalence and $\partial N \hookrightarrow \overline{M - N} - D$ is an $(\varepsilon, 1)$-equivalence in $\overline{M - N} - C$.

We are now ready for the last part of the argument. We want to prove that for every $\varepsilon_1 > \varepsilon_2 > \cdots$, $\lim \varepsilon_i = 0$, there are neighborhoods $N_1 \supset\supset N_2 \supset\supset \cdots$ of ∞, $\cap N_i = \varnothing$, so that $\partial N_i \hookrightarrow N_i - \mathring{N}_{i+1}$ and $\partial N_{i+1} \hookrightarrow N_i - \mathring{N}_{i+1}$ are $(\varepsilon_i, 1)$-equivalences. We will only give the details for the construction of N_1 and N_2, as this will indicate how the construction goes in general. Choose any $\varepsilon_1', \varepsilon_1'' > 0$ and start by choosing N_1

to be a neighborhood of ∞ for which $\partial N_1 \hookrightarrow N_1$ is an $(\varepsilon_1', 1)$-equivalence and ∂N_1 $\hookrightarrow \overline{M - N_1} - D$ is an $(\varepsilon_1', 1)$-equivalence in $\overline{M - N_1} - C$, for some compacta $C \subset D$ $\subset M - N_1$. Now choose another neighborhood $N_2 \subset\subset N_1$ of ∞ so that $\partial N_2 \hookrightarrow N_2$ is an $(\varepsilon_1'', 1)$-equivalence and $\partial N_2 \hookrightarrow N_1 - \mathring{N}_2$ is an $(\varepsilon_1', 1)$-equivalence in $(N_1 - \mathring{N}_2) \cup$ $(M - D)$. We can do this since our choice of N_1 can be made as close to ∞ as we want. It is easy to see that $\partial N_1 \hookrightarrow N_1 - \mathring{N}_2$ is an $\bar{\varepsilon}_1$-equivalence, and by the same argument it also follows that $\partial N_2 \hookrightarrow N_1 - \mathring{N}_2$ is an $\bar{\varepsilon}_1$-equivalence, where the size of $\bar{\varepsilon}_1$ depends on the size of $\varepsilon_1' + \varepsilon_1''$. So all we have to do is choose ε_1', ε_1'' so that $\bar{\varepsilon}_1 \leqslant \varepsilon_1$, and we are now ready for the next step of the construction.

II. $(m - 3)$-*Neighborhoods of* ∞. In this step we want to show that the N_i of step I can be chosen to additionally satisfy $\partial N_i \hookrightarrow N_i$ is $(m - 3, \varepsilon_i)$-connected, i.e., $(N_i, \partial N_i)$ is $(m - 3, \varepsilon_i)$-connected. The usual procedure is to inductively work up to the integer $m - 3$ by showing that the N_i can be constructed so that $(N_i, \partial N_i)$ is (k, ε_i)-connected, for $2 \leqslant k \leqslant m - 3$. The case $k = 2$ is the easiest to deal with. Given $\{N_i\}$ as in step I it will suffice to show how to rechoose N_1 to obtain N_1' for which $\partial N_1' \hookrightarrow N_1'$ is $(\varepsilon_1', 2)$-connected, for ε_1' a number whose size depends on the sizes of the ε_i. By Proposition 5.1 we can attach a finite number of 2-cells to ∂N_1 to form $K \subset N_1 - N_2$ so that $\partial N_1 \subset K$ and $K \hookrightarrow N_1$ is $(\varepsilon, 2)$-connected, where ε depends on ε_1. Since $\partial N_1 \hookrightarrow N_1 - \mathring{N}_2$ is an $(\varepsilon_1, 1)$-equivalence we may choose the cells of $K - \partial N_1$ to be trivially attached, and we may choose these cells so that their images in B are small. Let R be a regular neighborhood of $\overline{M - N_1} \cup K$ and let $N_1' = M - \mathring{R}$. Then $N_1' \supset\supset N_2 \supset\supset \cdots$ is a new sequence of neighborhoods of ∞. To see that $\partial N_1' \hookrightarrow N_1'$ $- \mathring{N}_2$ is an $(\varepsilon_1', 1)$-equivalence we use the fact that $K \hookrightarrow N_1$ is an $(\varepsilon, 1)$-equivalence along with general positioning to conclude that $\partial N_1' \hookrightarrow N_1'$ is an $(\varepsilon_1'', 1)$-equivalence, and this easily implies that $\partial N_1' \hookrightarrow N_1' - \mathring{N}_2$ is an $(\varepsilon_1', 1)$-equivalence. Since $\partial N_2 \hookrightarrow N_1$ $- \mathring{N}_2$ is an $(\varepsilon_1, 1)$-equivalence we can similarly conclude that $\partial N_2 \hookrightarrow N_1' - \mathring{N}_2$ is an $(\varepsilon_1', 1)$-equivalence. Finally we again can use general positioning to prove that $(N_1', \partial N_1')$ is $(\varepsilon_1', 2)$-connected simply because $m \geqslant 6$. (Note that this is the last time that we will be able to use general positioning to show that $(N_1', \partial N_1')$ is (ε_1', k)-connected because the dimension restriction $m \geqslant 6$ is now critical.) This completes the construction of N_1' and therefore the proof of the case $k = 2$.

The cases $3 \leqslant k \leqslant m - 3$ are treated somewhat like $k = 2$ above except that a new problem arises. Given N_1 for which $(N_1, \partial N_1)$ is $(\varepsilon_1, k - 1)$-connected we can use Proposition 5.1 to form K by attaching a finite number of k-cells to ∂N_1, and we can extend $\partial N_1 \hookrightarrow N_1$ to an (ε, k)-connected map $f: K \to N_1$, where the size of ε depends on the size of ε_1. The new problem we encounter is that in general we cannot embed K in N_1 because of dimension restrictions. However the Whitney trick, along with the $(\varepsilon_1, k - 1)$-connectivity of $(N_1, \partial N_1)$, does assure us that K can be embedded as desired. Of course the heart of the Whitney trick is the existence of 2-discs which are then used to construct certain isotopies. In order for these isotopies to be small in B these 2-discs must have small diameters in B. To achieve this one works locally, i.e., over open subsets of B of small diameters, thus forcing these 2-discs to have small diameters in B. This is all explained in [7, §7.3]. Thus K can be embedded in $N_1 - N_2$ so that $K \hookrightarrow N_1$ is (ε, k)-connected. Now the proof proceeds as in the case $k = 2$. We define $N_1' = M - \mathring{R}$, where R is a regular neighborhood of

$M - N_1 \cup K$. Then $N_1' \supset\supset N_2 \supset\supset \cdots$ is a new sequence of neighborhoods of ∞ such that $\partial N_1' \hookrightarrow N_1' - \mathring{N}_2$ and $\partial N_2 \hookrightarrow N_1' - \mathring{N}_2$ are $(\varepsilon_1', 1)$-equivalences (by general positioning). To see that $(N_1', \partial N_1')$ is (ε_1', k)-connected we use the fact that (N_1, K) is (ε, k)-connected along with excision and our Controlled Hurewicz Theorem, Theorem 3.1 (cf. [7, p. 318]).

III. $(m - 2)$-*Neighborhoods of* ∞. It follows from step II above that for any $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ there are neighborhoods $N_1 \supset\supset N_2 \supset\supset \cdots$ of ∞, $\cap N_i = \varnothing$, so that $(N_i, \partial N_i)$ is $(\varepsilon_i, m - 3)$-connected and $\partial N_{i+1} \hookrightarrow N_i - \mathring{N}_{i+1}$ is an $(\varepsilon_i, 1)$-equivalence. For any $\varepsilon > 0$ we will show that there exists an i and a neighborhood $N \subset N_i$ of ∞ for which $(N, \partial N)$ is $(\varepsilon, m - 2)$-connected and $N \hookrightarrow N_i$ is an $(\varepsilon, 1)$-equivalence. Once we have established this it will follow that the sequence $\{N_i\}$ can be rechosen so that, additionally, $(N_i, \partial N_i)$ is $(\varepsilon_i, m - 2)$-connected.

We begin by describing the problem that we are facing. Let us assume that the N_i of step II exist, where $\lim \varepsilon_i = 0$. Choose i large and use Proposition 5.1 to extend $\partial N_i \hookrightarrow N_i$ to an $(\varepsilon, m - 2)$-connected map $f: K \to N_i$, where ε is small and $(K, \partial N_i)$ is a relative CW complex formed by attaching a finite number of $(m - 2)$-cells to ∂N_i. Of course the problem is now to represent the images of these $(m - 2)$-cells in N_i by embedded $(m - 2)$-cells, for once this is done they can be thickened to handles whose interiors can then be deleted from N_i to yield our desired N (cf. step II above). This representation problem is solved by relating K to a handlebody structure on N_i, and then making use of Theorem 5.3 to alter this handlebody structure so that the $(m - 2)$-cells are represented by a subset of the handles.

By the tameness at ∞ we can choose $j > i$ large enough so that there is a homotopy of N_i into $N_i - \mathring{N}_j$ rel ∂N_i, where the image of this homotopy in B is small. As in [7, §7.4] we may assume that the cobordism $(N_i - \mathring{N}_j, \partial N_i)$ consists only of $(m - 3)$- and $(m - 2)$-handles with small diameters in B. Let W be the union of ∂N_i and the $(m - 3)$-handles, and let $V = N_i - \mathring{N}_j$. From a homotopy point of view we may regard $(V, \partial N_i)$ as a relative CW complex which is formed by attaching $(m - 3)$- and $(m - 2)$-cells to ∂N_i. Using the $(\varepsilon_i, m - 3)$-connectivity of $(N_i, \partial N_i)$ we can find a map $r: N_i \to N_i$ for which $r(W) = \partial N_i$, $r | \partial N_i = \mathrm{id}$, and $r \simeq \mathrm{id}$ rel ∂N_i via a $p^{-1}(\varepsilon_i)$-homotopy. Proceeding as in the proof of Proposition 5.1 we can define our CW complex K to be $V/\!\sim$, where $\sim$ is the equivalence relation generated by $x \sim r(x)$ for $x \in W$. Let $q: V \to K$ be the resulting quotient map. The map $f: K \to N_i$ is then defined by using levels of the homotopy $r \simeq \mathrm{id}$. Also let $s: N_i \to V$ be a map for which $s | \partial N_i = \mathrm{id}$ and $s \simeq \mathrm{id}$ rel ∂N_i via a homotopy whose image in B is small. Then $g = qs: N_i \to K$ is a map, and it is easy to see that $fg \simeq \mathrm{id}$ rel ∂N_i (with small control in B), i.e., f is an ε-domination with inverse g. Of course, f is also an $(\varepsilon, m - 2)$-connected map.

The next step is to use Theorem 5.3. Since p is nice at ∞ it is possible to find a compact relative CW complex $(P, \partial N_i)$ and an ε-equivalence $P \simeq N_i$ rel ∂N_i. This can be done for every $\varepsilon > 0$ without the size of $\dim P$ increasing. (See Remark 2 following Theorem 2.3 of [2], and see the proof of the Controlled Boundary Theorem of [3].) Assume that the $(m - 2)$-cells of V are based at (x_i) in ∂N_i, and so the $(m - 2)$-cells of K are also based at (x_i). Let $A: (x_i) \to (x_i)$ be the ε-path matrix resulting from the map $gf: K \to K$. The conclusion of Theorem 5.3 requires that A

and $I - A$ be stabilized by identity matrices. $I - A$ can be stabilized by adding cancelling pairs of $(m - 3)$- and $(m - 2)$-handles to ∂N_i. A can be stabilized by adding cancelling pairs of $(m - 3)$- and $(m - 2)$-handles to ∂N_i, and then deleting from N_i the interiors of the $(m - 3)$-handles. (This is exactly the situation encountered in [7, §7.4].) Assuming that such stabilizations have been carried out, Theorem 5.3 gives us an ε-path isomorphism $B : (x_i) \to (x_i)$ so that AB and $(I - A)B$ are *geometric*. By this we mean that we can write $(x_i) = ((y_j), (z_k))$ so that the columns of AB corresponding to the elements z_k are all 0, and the columns of $(I - A)B$ corresponding to the elements y_j are all 0. We now interpret this data geometrically in (V, W).

It is easy to see that $fq : (V, W) \to (N_i, \partial N_i)$ is homotopic to $r| : (V, W) \to (N_i, \partial N_i)$. Thus our problem of representing the images of the $(m - 2)$-cells of K in N_i by embedded cells is equivalent to that of representing the images of the $(m - 2)$-cells of V in N_i (under the map r) by embedded cells. Since $qs = g$ we have $qsr \simeq qsfq = gfq$. But the path matrix of q is the identity, so the path matrix of $sr : (V, W) \to (V, W)$ is just A. The next step is to dispose of B. According to the matrix identities on pp. 35–36 of [8] we can use B to rechoose the handlebody structure of V so that the columns of A corresponding to the z_k are 0, and the columns of $I - A$ corresponding to the y_j are 0. This can be interpreted homotopically as follows. Let

$$\phi_j : \left(I^{m-2}, \partial I^{m-2}, * \right) \to \left(V, W, y_j \right)$$

be a characteristic map for the $(m - 2)$-cells of V based at y_j and similarly let ψ_k be a characteristic map for the $(m - 2)$-cells based at z_k. Then we have homotopies $sr\phi_j \simeq \phi_j$ and $sr\psi_k \simeq *$ (constant map), where these homotopies are small in B. The first homotopy implies that each map $\phi_j| : (\partial I^{m-2}, *) \to (W, y_j)$ is homotopic to the constant map. The second homotopy implies that only the maps $r\phi_j$ have to be represented by embeddings into N_i.

Now using the argument of [7, §7.4] each $\phi_j| : \partial I^{m-2} \to W$ (which is an embedding) is the 0-level of an embedding of $\partial I^{m-2} \times [0, 1]$ into W so that the 1-level goes into ∂N_i. Thus the map ϕ_j extends to a characteristic map $\tilde{\phi}_j$ for an $(m - 2)$-cell which is attached to ∂N_i. The $\tilde{\phi}_j$ are embeddings and they have disjoint images for different j. It is easy to check that $r\phi_j \simeq \tilde{\phi}_j$, for each j, and so our representation problem is solved. Thus N can be constructed as desired.

IV. *Conclusion of Proof.* It follows from step III above that for any $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ there are clean neighborhood $N_1 \supset\supset N_2 \supset\supset \cdots$ of ∞, $\bigcap N_i = \varnothing$, so that $\partial N_i \hookrightarrow N_i$ is $(\varepsilon_i, m - 2)$-connected and $\partial N_{i+1} \hookrightarrow N_i - \mathring{N}_{i+1}$ is an $(\varepsilon_i, 1)$-equivalence. Since each $N_i - \mathring{N}_{i+1}$ can be given a handlebody structure with handles of only indices $m - 3$ and $m - 2$ whose images in B are small, it easily follows from the tameness at ∞ that $\partial N_i \hookrightarrow N_i$ is an ε_i'-equivalence, where $\lim \varepsilon_i' = 0$. By a local duality argument (as in [7, §5]) we conclude from Theorem 3.1 that $\partial N_j \hookrightarrow N_i - \mathring{N}_j$ is an ε_i''-equivalence, for any i, where $\lim \varepsilon_i'' = 0$. (This of course requires that the N_i be chosen so that $\partial N_j \hookrightarrow N_i - \mathring{N}_j$ is an $(\varepsilon_i, 1)$-equivalence, for every $j > i$.) So summarizing we might as well assume that $\partial N_i \hookrightarrow N_i$ and $\partial N_j \hookrightarrow N_i - \mathring{N}_j$ are ε_i-equivalences, where $\lim \varepsilon_i = 0$. We will use this information to show that M admits a controlled boundary.

Since p is nice at ∞ it follows that for every $\varepsilon > 0$ there exists a $\delta > 0$ and an i_0 so that $p: N_i \to B$ is (ε, δ)-nice, for all $i \geqslant i_0$. Thus for any given decreasing set $\{\delta_j\}_{j=1}^k$ of positive numbers we can find an $i_1 \geqslant i_0$ large enough so that $p: N_i \to B$ is (δ_j, δ_{j+1})-nice, for all j and all $i \geqslant i_1$. Observe that by a sufficiently large choice of i_1 we can choose these δ_j's small enough so that the Controlled h-Cobordism applies (see Addendum 2). So using the Controlled h-Cobordism Theorem it follows that for every $\varepsilon > 0$ there exists an i such that each cobordism $(N_j - \mathring{N}_j, \partial N_j)$ is an ε-product, for all $j \geqslant i + 1$. Now select $i_1 < i_2 \cdots$ so that $(N_{i_n} - \mathring{N}_{i_{n+1}}, \partial N_{i_n})$ is an ε_n-product, where the sum $\Sigma \varepsilon_n$ is finite. Then piece together these ε_n-product structures to obtain a homeomorphism $h: \partial N_{i_1} \times [0, 1) \to N_{i_1}$ such that $h(x, 0) = x$. By the finite condition it follows that ph extends continuously to a map of $\partial N_1 \times [0, 1]$ to B. Thus M admits a controlled boundary. $\square$

7. Proof of the controlled h-cobordism theorem. In [9] a proof is given of the proper s-cobordism theorem which uses a splitting technique to carve up a proper h-cobordism into a countable union of compact h-cobordisms. In our proof of the Controlled h-Cobordism Theorem we will use this splitting technique to carve up a controlled h-cobordism into a finite union of small h-cobordisms so that the usual uncontrolled s-cobordism theorem can then be applied to each of these small cobordisms. It will be helpful to recall the splitting technique that was used in [9], but first some notation. If (X, Y) is a compact polyhedral pair and $p: X \to B$ is a controlling map to our parameter space, then the symbol $X \overset{\varepsilon}{\wedge} Y$ means that there exists a compact polyhedron K containing X and Y as subpolyhedra and PL retractions $\alpha: K \to X$, $\beta: K \to Y$ with contractible point-inverses such that $\alpha \simeq \beta$ rel Y via a $p^{-1}(\varepsilon)$-homotopy.

ONE-DIMENSIONAL SPLITTING LEMMA. *For every $\varepsilon > 0$ and integer m there exists a $\delta > 0$ so that if $(W^{m+1}, \partial_0 W) \overset{p}{\to} [0, 3]$ is a compact (δ, h)-cobordism and $W \overset{\delta}{\wedge} \partial_0 W$, then there exists a clean submanifold V of W for which*

(1) $p^{-1}([0, 1]) \subset V \subset p^{-1}([0, 2])$,
(2) $(V, V \cap \partial_0 W) \to [0, 3]$ *is an (ε, h)-cobordism,*
(3) $(\overline{W - V}, \overline{W - V} \cap \partial_1 W) \to [0, 3]$ *is an (ε, h)-cobordism.*

(In this statement we use $\partial_1 W$ for $\overline{\partial W - \partial_0 W}$, and it should be noted that the parameter space is $[0, 3]$.)

Very briefly the above result is established by using the methods of the proof of the boundary theorem of [8] to obtain the splitting. As explained in [9, p. 492] we can carry this out provided that given any clean submanifold V of W for which

$$p^{-1}([0, 1.5 - \gamma]) \subset V \subset p^{-1}([0, 1.5])$$

and γ is small, then the inclusion $\overline{\partial V - \partial_0 W} \hookrightarrow V$ can be extended to a homotopy equivalence by the addition of a compact polyhedron to

$$\overline{\partial V - \partial_0 W} \cap p^{-1}([1.5 - 2\gamma, 1.5]).$$

But for δ sufficiently small the condition $W \overset{\delta}{\wedge} \partial_0 W$ guarantees that this can be done.

More generally if the property $W \overset{\delta}{\wedge} \partial_0 W$ is not satisfied, then the obstruction to finding such an extension is a relative Wall finiteness obstruction as appears in the first exact sequence of [9] for analyzing infinite simple types.

In order to adapt the above ideas to our situation in which the parameter space is more complicated we will have to sharpen the above definition. For (X, Y) and $p : X \to B$ as above, the symbol. $X \overset{\epsilon}{\underset{n}{\wedge}} Y$ means that there exists K, α and β as above for which dim $K \leqslant n$. Now for any B let $\phi : B \to [0, 3]$ be a map. Here is our general splitting result.

GENERAL SPLITTING LEMMA. *For any B let $\epsilon > 0$ and n be given. There exists a $\delta > 0$ so that if $(W, \partial_0 W) \overset{p}{\to} B$ is a compact (δ, h)-cobordism and $W \overset{\delta}{\underset{n}{\wedge}} \partial_0 W$, then there exists a clean submanifold V of W for which*
   (1) $(\phi p)^{-1}([0, 1]) \subset V \subset (\phi p)^{-1}([0, 2])$,
   (2) $(V, V \cap \partial_0 W) \overset{p}{\to} B$ *is an (ϵ, h)-cobordism*,
   (3) $(\overline{W - V}, \overline{W - V} \cap \partial_1 W) \overset{p}{\to} B$ *is an (ϵ, h)-cobordism.*

In analogy with the One-Dimensional Splitting Lemma this result is established by using the methods of proof of our Controlled Boundary Theorem. Note that we are not guilty of circular reasoning, for we use only the first part of the proof which did not make use of the Controlled h-Cobordism Theorem. It should be clear why the controlling integer n is needed, for it was also needed in the proof of the Controlled Boundary Theorem to guarantee the small ϵ-control in B.

There is one more ingredient we will need for the proof of the main result of this section. It is the Approximation Theorem from [2].

APPROXIMATION THEOREM [2]. *Given integers m and b there exists an integer n which satisfies the following property: for every B and $\gamma > 0$ there exists a decreasing set $\{\gamma_i\}_{i=1}^{l}$ of positive numbers so that if (X, Y) is a compact polyhedral pair, $p : X \to B$ is (γ_i, γ_{i+1})-nice for each i, dim $X \leqslant m$ and dim $B \leqslant b$, and $Y \hookrightarrow X$ is a γ_i-equivalence, then $X \overset{\gamma}{\underset{n}{\wedge}} Y$.*

REMARKS. 1. It is important to note that l depends only on dim B, and the γ_i can be rechosen in the same manner that the δ_i of the Controlled h-Cobordism can be rechosen (see Addendum 2 of §1).

2. There is also a relative version of the above result, where the assumption that p be (γ_i, γ_{i+1})-nice is replaced by the somewhat weaker condition that there exists a decreasing sequence $\{X_i\}_{i=1}^{l}$ of closed subpolyhedra of X for which
   (1) $X_l \cap Y \hookrightarrow X_l$ is a γ_l-equivalence,
   (2) $X - X_l \subset Y$,

(3) for each $b \in B$ and i, the inclusion

$$\left(p^{-1}\left(S_{\gamma_{i+1}}(b)\right) \cap X_{i+1} \right) \times T^k \hookrightarrow \left(p^{-1}\left(S_{\gamma_i}(b)\right) \cap X_i \right) \times T^k$$

induces the 0-homomorphism on Whitehead groups, for all k. (We say that p is (γ_i, γ_{i+1})-nice on (X_i, X_{i+1}).)

We are now ready to begin the proof of the Controlled h-Cobordism Theorem. It will be convenient to first assume that B is a compact polyhedron, $\dim B = b$. For our given $m + 1$ we have the integers n and l of the Approximation Theorem. For our choice of k we will use $k = lb$. We will show that if the $\{\delta_i\}_{i=1}^k$ are chosen appropriately and $p : W^{m+1} \to B$ is (δ_i, δ_{i+1})-nice, then $(W, \partial_0 W)$ is an ε-product. For notation choose a subdivision of B which is fine enough so that all the simplices have small diameters with respect to ε. $B^{(r)}$ denotes the rth barycentric subdivision of B and B^i denotes the i skeleton of B. Also $\mathrm{St}(B^i, B^{(r)})$ is the star of B^i in the rth barycentric subdivision, which we abbreviate as B_r^i.

Now for the first step we can use the Approximation Theorem and the General Splitting Lemma to find a clean submanifold W_{b-1} of W for which

(1) $B_5^{b-1} \subset p(W_{b-1}) \subset B_2^{b-1}$,

(2) $(W_{b-1}, W_{b-1} \cap \partial_0 W) \to B$ and $(\overline{W - W_{b-1}}, \overline{W - W_{b-1}} \cap \partial_1 W) \to B$ are $(\delta_{l(b-1)}, h)$-cobordisms.

This requires that the number $\delta_{l(b-1)}$ must correspond to the number γ of the Approximation Theorem, and the set $\{\delta_i\}_{i=l(b-1)+1}^k$ must correspond to the set $\{\gamma_i\}_{i=1}^l$. For convenience we may assume that p is PL and each $p^{-1}(B_r^{b-1})$ is clean. Thus the above splitting can be trivially extended to additionally satisfy the property that $W_{b-1} \cap p^{-1}(\overline{B_2^{b-1} - B_3^{b-1}})$ is a small collar on $p^{-1}(\overline{B_2^{b-1} - B_3^{b-1}}) \cap \partial_0 W$ and $\overline{W - W_{b-1}} \cap p^{-1}(\overline{B_4^{b-1} - B_5^{b-1}})$ is a small collar on $p^{-1}(\overline{B_4^{b-1} - B_5^{b-1}}) \cap \partial_1 W$. If $\{\Delta_i^b\}$ is the collection of b-cells in B, then the h-cobordism $(\overline{W - W_{b-1}}, \overline{W - W_{b-1}} \cap \partial_1 W)$ is a disjoint union of small h-cobordisms $(V_i, V_i \cap \partial_1 W)$, where $V_i = p^{-1}(\Delta_i^b) \cap \overline{W - W_{b-1}}$. By the relative version of the Approximation Theorem we conclude that the Whitehead torsion of the h-cobordism $(V_i, V_i \cap \partial_1 W)$ is 0 (this is the reason why the above splitting was extended). Now extend $(W_{b-1}, W_{b-1} \cap \partial_0 W)$ to a $(\delta_{l(b-1)}, h)$-cobordism $(\tilde{W}_{b-1}, \partial_0 W)$ by adding to W_{b-1} a small collar on $\partial_0 W$. The triviality of the h-cobordism $(V_i, V_i \cap \partial_1 W)$ gives us a homeomorphism h_1 of W onto $\tilde{W}_{b-1}$ which is supported on $p^{-1}(\overline{B - B_5^{b-1}})$.

This completes the first step of our construction which succeeds in replacing $(W, \partial_0 W)$ by $(\tilde{W}_{b-1}, \partial_0 W)$, where $\tilde{W}_{b-1}$ contains the portion of W that lies over some small regular neighborhood of B^{b-1}, and $\tilde{W}_{b-1}$ is a small collar on $\partial_0 W$ over the complement of some larger regular neighborhood of B^{b-1}. The next step is to replace $(\tilde{W}_{b-1}, \partial_0 W)$ by $(\tilde{W}_{b-2}, \partial_0 W)$, where $\tilde{W}_{b-2}$ contains the portion of W that lies over some small regular neighborhood of B^{b-2} and $\tilde{W}_{b-2}$ is a small collar on $\partial_0 W$ over the complement of some larger regular neighborhood of B^{b-2}. This is accomplished by applying the above splitting ideas to the cobordism $(\tilde{W}_{b-1}, \partial_0 W)$. We then obtain a homeomorphism $h_2 : \tilde{W}_{b-1} \to \tilde{W}_{b-2}$ which is the identity over a regular neighborhood of B^{b-2} and over the complement of a regular neighborhood of B^{b-1}. More specifically, the support of h_2 lies over a small regular neighborhood

of the set of barycenters of the $(b - 1)$-simplices of B. This completes the second step of the construction.

Now continuing the above process we can construct controlled h-cobordisms $(\tilde{W}_i, \partial_0 W)$ and homeomorphisms

$$W \overset{h_1}{\to} \tilde{W}_{b-1} \overset{h_2}{\to} \tilde{W}_{b-2} \to \cdots \overset{h_b}{\to} \tilde{W}_0,$$

where $(\tilde{W}_0, \partial_0 W)$ is a small collar on $\partial_0 W$ over the complement of some regular neighborhood of B^0. Thus $(\tilde{W}_0, \partial_0 W)$ is essentially a union of small h-cobordisms—one for each vertex of B. By the relative version of the Approximation Theorem each one of these h-cobordisms is trivial. Thus we have a homeomorphism $k : \tilde{W}_0 \to \partial_0 W \times [0, 1]$ and $h = k h_b \cdots h_1 : W \to \partial_0 W \times [0, 1]$ is a homeomorphism as desired. It is clear that if the regular neighborhoods are appropriately chosen, then we must have ε-control on h because for each point x of W we must have $p(x)$ and $ph(x)$ in the star of some simplex. This completes the proof of the polyhedral case.

The proof of the general case is just an easy reduction to the polyhedral case. For B a finite-dimensional compact metric space choose $B \subset B'$, where B' is a compact polyhedron, and let $p' : W \to B'$ be the composition $W \overset{p}{\to} B \hookrightarrow B'$. Then one easily shows that the above case, which deals with the controlled h-cobordism $(W, \partial_0 W) \overset{p'}{\to} B'$, implies that $(W, \partial_0 W) \overset{p}{\to} B$ is an ε-product. It should be mentioned that if k' is the integer which described how many of the δ_i are needed for B', then $k = 3k' - 2$ describes how many of the δ_i are needed for B (cf. the proof of Lemma 7.1 of [2]).

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